# Permutability graphs of subgroups of some finite non-abelian groups

R. Rajkumar[*], P. Devi[†]

Department of Mathematics, The Gandhigram Rural Institute – Deemed University, Gandhigram – 624 302, Tamil Nadu, India

**Abstract**

In this paper, we study the structure of the permutability graphs of subgroups, and the permutability graphs of non-normal subgroups of the following groups: the dihedral groups $D_n$, the generalized quaternion groups $Q_n$, the quasi-dihedral groups $QD_{2^n}$ and the modular groups $M_{p^n}$. Further, we investigate the number of edges, degrees of the vertices, independence number, dominating number, clique number, chromatic number, weakly perfectness, Eulerianness, Hamiltonicity of these graphs.

**Keywords and phrases:** Permutability graphs, Non-abelian groups, Independence number, dominating number, Weakly perfect, Eulerian, Hamiltonian.

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## 1 Introduction

One can study the properties of an algebraic structure by associating a suitable graph with it and by using of the tools of graph theory. In recent years this has been a topic of interest among algebraic graph theorists and they have contributed significantly; in particular, when the algebraic structure is a group (see, for example [1], [2], [3], [13]).

[*]e-mail: rrajmaths@yahoo.co.in

[†]e-mail: pdevigri@gmail.com

M. Bianchi, A. Gillio and L. Verardi in [4], defined a graph corresponding to a group $G$, called the *permutability graph of non-normal subgroups of* $G$ having all the proper non-normal subgroups of $G$ as its vertices and two vertices $H$ and $K$ are adjacent if $HK = HK$; equivalently $HK$ is a subgroup of $G$. We denote this graph by $\Gamma_N(G)$. They focused on finding the number of connected components and the diameter of this graph. Further results on these graphs can be found in [5], [7].

In [10], the authors considered the general setting as follows: For a group $G$, *the permutability graph of subgroups of* $G$, denoted by $\Gamma(G)$, is a graph with vertex set consists of all the proper subgroups of $G$ and two vertices in $\Gamma(G)$ are adjacent if the two corresponding subgroups permute in $G$. There in we have studied the planarity of these graphs. Further properties of these graphs like bipartiteness, completeness and freeness of these graphs from some class of graphs were investigated in [11].

The aim of this paper is to study the structure and properties of the permutability graphs of subgroups, and the permutability graphs of non-normal subgroups of finite non-abelian groups. Especially, we consider the dihedral groups $D_n$, the generalized quaternion groups $Q_n$, the quasi-dihedral groups $QD_{2^n}$ and the modular groups $M_{p^n}$. Even though the subgroup structure of these groups are well known, we particulary focus on how much information about the subgroup permutability of these groups can be expressed in terms of the graph theoretic properties of their corresponding permutability graphs.

The rest of this paper is arranged as follows: In Section 2, we introduce some basic definitions and notations that we will use in this article. In Section 3, for a given group $G$, we present some basic results which gives the relationship between $\Gamma_N(G)$ and $\Gamma(G)$. In Section 4, we consider the dihedral groups $D_n$ and study the structure and properties of $\Gamma_N(D_n)$ and $\Gamma(D_n)$. In particular, we give the degrees of the vertices, number of edges, independence number, dominating number, chromatic number, clique number, weakly perfectness, Eulerianness of these two graphs. Also we investigate Hamiltonicity of $\Gamma_N(D_n)$ and $\Gamma(D_n)$ for some values of $n$. In Sections 5, 6 and 7 we investigate the same for the generalized quaternion groups $Q_n$, the quasi-dihedral groups $QD_{2^n}$ and the

modular groups $M_{p^n}$ respectively. In the last section, we conclude with some problems of further research.

## 2 Preliminaries and Notations

For a simple graph $G$, we denote its vertex set and edge set by $V(G)$ and $E(G)$ respectively. A graph is *complete* if all the vertices are adjacent with each other. A complete graph on $n$ vertices is denoted by $K_n$. An *independent set* is a set of vertices in $G$ of which no two are adjacent. An independent set is *maximal* if it is not a proper subset of any independent set of G. The *independence number* $\alpha(G)$ of $G$ is the cardinality of a largest maximal independent set in $G$. A set $D$ of vertices in $G$ is a *dominating set* in $G$ if every vertex in $G$ is contained in or is adjacent to a vertex in $D$. The *dominating number* $\gamma(G)$ of $G$ is the cardinality of a smallest dominating set in $G$. A *clique* is a set of vertices in $G$ such that any two are adjacent. The *clique number* $\omega(G)$ of $G$ is the cardinality of a largest clique in $G$. The *chromatic number* $\chi(G)$ of $G$ is the smallest number of colours needed to colour the vertices of $G$ such that no two adjacent vertices gets the same colour. A graph $G$ is said to be *weakly perfect* if $\omega(G) = \chi(G)$. The $\deg_G(v)$ of a vertex $v$ of a graph $G$ is the number of edges incident with $v$. A graph $G$ is said to be *$r$-partite* if the vertex of $G$ can be partitioned into $r$ sets such that no two vertices in each partition are adjacent. $G$ is *complete $r$-partite* if every vertex in each partition is adjacent with all the vertices in the remaining partition.

A *walk* joining two vertices $v_0$ and $v_n$ in $G$ is an alternating sequence of vertices and edges $v_0$, $e_1$, $v_1$, $e_2$, $\ldots$ ,$v_{n-1}$, $e_n$, $v_n$ beginning and ending with vertices such that each edge $e_i$ is incident with $v_{i-1}$ and $v_i$. A walk is a *path* if all its vertices are distinct. We denote a path joining two vertices $u$ and $v$ in $G$ simply, as $(u =)v_0 - v_1 - v_2 - \cdots - v_n(= v)$ with the understanding that there is an edge joining $v_{i-1}$ and $v_i$, for each $i$, $1 \leq i \leq n$. A walk is called *closed* if its initial and terminal vertices coincides. A closed walk in which all the vertices are distinct is a *cycle*. $G$ is said to be *connected* if any distinct two vertices are joined by a path. A *component* of a graph $G$ is a maximal connected subgraph of $G$.

A number of component of a graph $G$ is denoted by $c(G)$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. Their *union* $G_1 \cup G_2$ is a graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. Their *join* $G_1 + G_2$ is a graph consist of $G_1 \cup G_2$ together with all the lines joining points of $V_1$ to points of $V_2$. For any connected graph $G$, we write $nG$ for the graph with $n$ components each isomorphic to $G$. A graph $G$ is said to be *Eulerian*, if it contains a closed trail which contains every edge of $G$ exactly once; equivalently, $G$ is Eulerian if and only if every vertex in $G$ has even degree. A graph is said to be *Hamiltonian*, if it contains a cycle having all the vertices of the graph. A *split graph* is a graph in which the vertices can be partitioned into a clique and an independent set. For basic graph theory terminology, we refer to [14].

We recall the following theorem which will be use in the sequel.

**Theorem 2.1.** ([14, Proposition 7.2.3]) If a graph $G$ is Hamiltonian, then $c(G-S) \leq |S|$ for every non-empty subset $S$ of $V(G)$.

For any positive integer $n$, $\tau(n)$ denotes the number of divisors of $n$ and $\sigma(n)$ denotes the sum of all divisors of $n$. A positive integer $n$ is said to be *deficient* if $\sigma(n) < 2n$.

**Note 2.1.** *In some of the results of this paper, we will use the following basic facts which can be found in any basic number theory book; for instance, see [9]: Let* $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ *be a integer, where* $p_i$*'s are distinct primes and* $\alpha_i \geq 1$. *Then* $\tau(n) = \prod_{i=1}^{k}(\alpha_i + 1)$. *Moreover,*

(i) $\tau(n)$ *is even if and only if* $\alpha_i$ *is odd for some* $i \in \{1, 2, \dots, k\}$.

(ii) $\tau(n)$ *is odd if and only if* $\alpha_i$ *is even for every* $i \in \{1, 2, \dots, k\}$.

(iii) *If* $n$ *is odd, then* $\sigma(n)$ *is odd (even) if and only if* $\tau(n)$ *is odd (even).*

(iv) *If* $n = 2^\alpha$, $\alpha \geq 1$, *then* $\sigma(n)$ *is odd.*

(v) *If* $n = 2^\alpha n'$, *with* $n' = p_2^{\alpha_2} \dots p_k^{\alpha_k}$ *is odd and* $\alpha \geq 1$, *then* $\sigma(n)$ *is odd (even) if and only if* $\tau(n')$ *is odd (even).*

# 3 Some basic results

Recall that a group in which all the subgroups are normal is known as a Dedekind group. For the characterization of Dedekind groups the reader can refer to [12, Theorem 5.3.7, p.143]. For a given group $G$, we can define $\Gamma_N(G)$ only when $G$ is other than a Dedekind group; and we can define $\Gamma(G)$ only when $G$ is other than the trivial group or a group of prime order. Also note that $|V(\Gamma(G)| = |L(G)| - 2$, where $L(G)$ denotes the subgroup lattice of $G$.

Since any normal subgroup of a group permutes with every other subgroups, so we have the following result.

**Theorem 3.1.** *Let $G$ be a finite non-simple group with $r$ proper normal subgroups. Then*

$$\Gamma(G) \cong \begin{cases} K_r + \Gamma_N(G), & \text{if } \ r \neq |L(G)| - 2; \\ K_r, & \text{otherwise.} \end{cases}$$

**Corollary 3.1.** *Let $G$ be a finite non-simple group with $r$ proper normal subgroups. If $r \neq |L(G)| - 2$, then the following holds:*

(i) $\deg_{\Gamma(G)}(H) = \begin{cases} |L(G)| - 3, & \text{if } H \text{ is normal in } G; \\ r + \deg_{\Gamma_N(G)}(H), & \text{otherwise.} \end{cases}$.

(ii) $|E(\Gamma(G))| = |E(\Gamma_N(G))| + \dfrac{r}{2}\ (2|L(G)| - r - 5)$.

(iii) $\alpha(\Gamma(G)) = \alpha(\Gamma_N(G))$.

(iv) $\omega(\Gamma(G)) = r + \omega(\Gamma_N(G))$.

(v) $\chi(\Gamma(G)) = r + \chi(\Gamma_N(G))$.

(vi) $\gamma(G) = 1$.

*Proof.* We prove part (ii) and the remaining parts of the result are immediate consequence of the previous theorem. Note that the proper number of non-normal subgroups of $G$ is

$|L(G)| - r - 2$. So by Theorem 3.1,

$$|E(\Gamma(G))| = \binom{r}{2} + r[|L(G)| - r - 2] + |E(\Gamma_N(G))|,$$

which leads to the result by simplification. □

# 4 Dihedral groups

The dihedral group of order $2n$ ($n \geq 3$) is defined by

$$D_n = \langle a, b \mid a^n = b^2 = 1, ab = ba^{-1} \rangle.$$

The subgroups of $D_n$ are listed below:

(i) cyclic groups $H_0^r := \langle a^{\frac{n}{r}} \rangle$ of order $r$, where $r$ is a divisor of $n$;

(ii) cyclic groups $H_i^1 := \langle ba^{i-1} \rangle$ of order 2, where $i = 1, 2, \ldots, n$;

(iii) dihedral groups $H_i^r := \langle a^{\frac{n}{r}}, ba^{i-1} \rangle$ of order $2r$, where $r$ is a divisor of $n$, $r \neq 1, n$ and $i = 1, 2, \ldots, \frac{n}{r}$.

The proper normal subgroups of $D_n$ are the subgroups $H_0^r$, $r \neq 1$ listed in (i), when $n$ is odd; the subgroups $H_0^r$, $r \neq 1$ listed in (i) and $H_i^{\frac{n}{2}}$, $i = 1, 2$ of index 2, when $n$ is even. Thus

$$|L(D_n)| = \tau(n) + \sigma(n), \tag{4.1}$$

and so

$$|V(\Gamma(D_n)| = \tau(n) + \sigma(n) - 2,$$

$$|V(\Gamma_N(D_n)| = \begin{cases} \sigma(n) - 1, & \text{if } n \text{ is odd;} \\ \sigma(n) - 3, & \text{otherwise.} \end{cases} \tag{4.2}$$

The following details about the permutability of subgroups of $D_n$ were given by Tărnăuceanu in [8, p. 2513-2516]. We will use these to prove some of our results in

this paper. Consider the subgroups $H_i^r$ and $H_j^s$, where $r$ and $s$ are the divisor of $n$, $i \in \{1, 2, \ldots, \frac{n}{r}\}$, $j \in \{1, 2, \ldots, \frac{n}{s}\}$. Then

$H_i^r H_j^s = H_j^s H_i^r$ if and only if $a^{2(i-j)} \in \langle a^{\frac{n}{[r,s]}} \rangle$, that is

$$\frac{n}{[r,s]} \,\Big|\, 2(i-j). \tag{4.3}$$

For a fixed divisor $r$ of $n$, and $i \in \{1, 2, \ldots, \frac{n}{r}\}$, let $x_i^r$ denotes the number of solutions of (4.3). The value of $x_i^r$ is described explicitly in the following cases:

**Case 1.** If $n$ is odd, then

$$x_i^r = \sum_{s|n} \frac{[r,s]}{s} = r \sum_{s|n} \frac{1}{(r,s)} \tag{4.4}$$

and so

$$\sum_{r|n} \sum_{i=1}^{\frac{n}{r}} x_i^r = g(n), \tag{4.5}$$

where $g$ is the function defined by $g(k) = k \sum_{r|k, s|k} \frac{1}{(r,s)}$, for all $k \in \mathbb{N}$. Then $g$ is a multiplicative function and

$$g(p^\alpha) = \frac{(2\alpha+1)p^{\alpha+2} - (2\alpha+3)p^{\alpha+1} + p + 1}{(p-1)^2} \tag{4.6}$$

for any prime $p$ and $\alpha \in \mathbb{N}$. If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $p_i$'s are distinct primes and $\alpha_i \geq 1$, then by (4.6)

$$g(n) = \prod_{i=1}^{k} \frac{(2\alpha_i+1)p_i^{\alpha_i+2} - (2\alpha_i+3)p_i^{\alpha_i+1} + p_i + 1}{(p_i-1)^2}. \tag{4.7}$$

**Case 2.** Let $n$ be even.

**subcase 2a.** If $n = 2^\alpha$, $\alpha \geq 2$, then

$$x_i^r = 2^{u+2} - 2u + 2\alpha - 3 \tag{4.8}$$

for every $r = 2^u$, where $0 \leq u \leq \alpha - 1$ and so

$$\sum_{r|n} \sum_{i=1}^{\frac{n}{r}} x_i^r = (\alpha - 1)2^{\alpha+3} + 9. \tag{4.9}$$

**subcase 2b.** If $n = 2^\alpha n'$, with $n'$ is odd, $\alpha \geq 1$, then for any divisor $r$ of $n$ with $r = 2^\beta r'$, where $\beta \leq \alpha$ and $r'|n'$,

$$x_i^r = \begin{cases} (2^{\alpha+1} - 1)x_i^{r'}, & \text{if } \beta = \alpha; \\ (2^{\beta+2} - 2\beta + 2\alpha - 3)x_i^{r'}, & \text{if } \beta < \alpha, \end{cases} \tag{4.10}$$

where $x_i^{r'}$ is given by (4.4) and so

$$\sum_{r|n} \sum_{i=1}^{\frac{n}{r}} x_i^r = [(\alpha - 1)2^{\alpha+3} + 9]g(n') \tag{4.11}$$

In addition to these cases, for any integer $n \geq 3$, it is easy to see that

$$x_1^n = \sigma(n) \tag{4.12}$$

and for each $i \in \{1, 2, \ldots, n\}$

$$x_i^1 = \tau(n). \tag{4.13}$$

If $n$ is even, then for each $i = 1, 2$

$$x_i^{\frac{n}{2}} = \sigma(n) \tag{4.14}$$

## 4.1 Properties of $\Gamma_N(D_n)$

In the following result, we describe the degrees of the vertices of $\Gamma_N(G)$.

**Theorem 4.1.** *Let $n \geq 3$ be an integer.*

(i) *If $n$ is odd, then $\deg_{\Gamma_N(D_n)}(H_i^r) = x_i^r - 2$, for each divisor $r$ of $n$, $r \neq n$, $i = 1, 2, \ldots, \frac{n}{r}$, where $x_i^r$ is given by (4.4).*

(ii) *Let $n$ be even. Then* $\deg_{\Gamma_N(D_n)}(H_i^r) = x_i^r - 4$*, for each divisor $r$ of $n$, $r \neq n, \frac{n}{2}$, $i = 1, 2, \ldots, \frac{n}{r}$, where $x_i^r$ is given by* (4.8) *if $n = 2^\alpha$, $\alpha \geq 2$; $x_i^r$ is given by* (4.10) *if $n = 2^\alpha n'$ with $n'$ is odd, $\alpha \geq 1$.*

*Proof.* One can observe that for each divisor $r$ of $n$ and $i \in \{1, 2, \ldots, \frac{n}{r}\}$, the number of dihedral subgroups of $D_n$ permutes with $H_i^r$ is $x_i^r$. So if $n$ is odd, then $\deg_{\Gamma_N(D_n)}(H_i^r) = x_i^r - |\{H_i^r, H_1^n\}| = x_i^r - 2$, where $x_i^r$ is given by (4.4). If $n$ is even, then $\deg_{\Gamma_N(D_n)}(H_i^r) = x_i^r - |\{H_i^r, H_1^n, H_1^{\frac{n}{2}}, H_2^{\frac{n}{2}}\}| = x_i^r - 4$, where $x_i^r$ is given by (4.8) and (4.10). □

**Corollary 4.1.** *Let $n \geq 3$ be an integer and $g$ denotes the arithmetic function given by* (4.7)*.*

(i) *If $n$ is odd, then*

$$|E(\Gamma_N(D_n))| = \frac{1}{2}\{g(n) - 3\sigma(n) + 2\}.$$

(ii) *If $n = 2^\alpha$, $\alpha \geq 2$, then*

$$|E(\Gamma_N(D_n))| = 2^\alpha(4\alpha - 11) + 14.$$

(iii) *If $n = 2^\alpha n'$, with $n'$ is odd, $\alpha \geq 1$, then*

$$|E(\Gamma_N(D_n))| = \frac{1}{2}\big\{[(\alpha - 1)2^{\alpha+3} + 9]g(n') - 7\sigma(n) + 12\big\}.$$

*Proof.* (i): If $n$ is odd, then by Theorem 4.1(i) and by using (4.5), (4.12), we have

$$\begin{aligned}
|E(\Gamma_N(D_n))| &= \frac{1}{2}\sum_{H \in \Gamma_N(D_n)} \deg_{\Gamma_N(D_n)}(H) \\
&= \frac{1}{2}\sum_{\substack{r|n \\ r \neq n}} \sum_{i=1}^{\frac{n}{r}} (x_i^r - 2) \\
&= \frac{1}{2}\Big[\sum_{r|n} \sum_{i=1}^{\frac{n}{r}} (x_i^r - 2) - (x_1^n - 2)\Big] \\
&= \frac{1}{2}\Big[\sum_{r|n} \sum_{i=1}^{\frac{n}{r}} x_i^r - 2\sum_{r|n} \sum_{i=1}^{\frac{n}{r}} 1 - (\sigma(n) - 2)\Big]
\end{aligned}$$

$$= \frac{1}{2}\Big[\sum_{r|n}\sum_{i=1}^{\frac{n}{r}} x_i^r - 2\sigma(n) - (\sigma(n)-2)\Big]$$
$$= \frac{1}{2}\big[g(n) - 3\sigma(n) + 2\big]$$

(ii)-(iii): By Theorem 4.1(ii) and by using (4.12) and (4.14), we have

$$\begin{aligned}
|E(\Gamma_N(D_n))| &= \frac{1}{2}\sum_{H\in\Gamma_N(D_n)} \deg_{\Gamma_N(D_n)}(H)\\
&= \frac{1}{2}\sum_{\substack{r|n\\ r\neq n,\frac{n}{2}}}\sum_{i=1}^{\frac{n}{r}}(x_i^r - 4)\\
&= \frac{1}{2}\Big[\sum_{r|n}\sum_{i=1}^{\frac{n}{r}}(x_i^r-4) - (x_1^n - 4) - \sum_{r=\frac{n}{2}}\sum_{i=1}^{\frac{n}{r}}(x_i^r-4)\Big]\\
&= \frac{1}{2}\Big[\sum_{r|n}\sum_{i=1}^{\frac{n}{r}} x_i^r - 4\sum_{r|n}\sum_{i=1}^{\frac{n}{r}} 1 - (\sigma(n)-4) - 2(\sigma(n)-4)\Big]\\
&= \frac{1}{2}\Big[\sum_{r|n}\sum_{i=1}^{\frac{n}{r}} x_i^r - 4\sigma(n) - (\sigma(n)-4) - 2(\sigma(n)-4)\Big]\\
&= \frac{1}{2}\Big[\sum_{r|n}\sum_{i=1}^{\frac{n}{r}} x_i^r - 7\sigma(n) + 12\Big]. \qquad (4.15)
\end{aligned}$$

Now to complete the proof it remains to consider the following cases:

**Case a.** If $n = 2^\alpha$, $\alpha \geq 2$, then (4.15) reduces to the result after simplification, by using (4.9) and $\sigma(n) = 2^{\alpha+1} - 1$.

**Case b.** If $n = 2^\alpha n'$, with $n'$ is odd and $\alpha \geq 1$, then (4.15) reduces to the result after simplification, by using (4.11). $\square$

In the following result, we determine the values of $n$ for which $\Gamma_N(D_n)$ is Eulerian.

**Corollary 4.2.** *Let $n \geq 3$ be an integer.*

(i) *If $n$ is odd with $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$, where $p_i$'s are distinct primes and $\alpha_i \geq 1$, then $\Gamma_N(D_n)$ is Eulerian if and only if $\alpha_i$ is odd for some $i \in \{1,2,\ldots,k\}$.*

(ii) *If $n = 2^\alpha$, $\alpha \geq 2$, then $\Gamma_N(D_n)$ is non-Eulerian.*

(iii) *If $n = 2^\alpha n'$, with $n' = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ is odd, where $p_i$'s are distinct primes and $\alpha \geq 1$, then $\Gamma_N(D_n)$ is Eulerian if and only if $\alpha_i$ is odd for some $i \in \{1, 2, \dots, k\}$.*

*Proof.* (i): By Theorem 4.1(i), $\deg_{\Gamma_N(D_n)}(H_i^r)$ is even if and only if $\displaystyle\sum_{s|n} \frac{[r,s]}{s}$ is even. Since $n$ is odd, $\dfrac{[r,s]}{s}$ is odd for each divisor $r$ and $s$ of $n$. So it follows that $\displaystyle\sum_{s|n} \frac{[r,s]}{s}$ is even only when $\tau(n)$ is even; this is true only when $\alpha_i$ is odd, for some $i \in \{1, 2, \dots, k\}$.
(ii): By Theorem 4.1(ii), $\deg_{\Gamma_N(D_n)}(H_i^r)$ is odd, for every $\alpha$ and so $\Gamma_N(D_n)$ is non-Eulerian.
(iii): By Theorem 4.1(ii), $\deg_{\Gamma_N(D_n)}(H_i^r)$ is even if and only if $x_i^r$ is even; that is when $x_i^{r'}$ is even, where $x_i^{r'}$ is given by (4.4). But for each divisor $r'$ of $n'$, $\frac{[r',s']}{s'}$ is always odd. So $x_i^{r'}$is even if and only if $\tau(n')$ is even; that is only when $\alpha_i$ is odd, for some $i$. □

Now we make further investigation to know more about the structure of $\Gamma_N(D_n)$. If $n$ is odd, then for each divisor $r$ of $n$, $r \neq n$, let $\mathcal{A}_r^o = \{H_i^r \mid i = 1, 2, \dots, \frac{n}{r}\}$. It is easy to see that $\mathcal{A}_r^o$ is a maximal independent set in $\Gamma_N(D_n)$. Also these sets are mutually disjoint and

$$\Gamma_N(D_n) = \bigcup_{\substack{r|n \\ r\neq n}} \mathcal{A}_r^o.$$

Also note that the number of such $A_r^o$ is $\tau(n) - 1$.
Let $n = 2^\alpha n'$ be even, with $n'$ is odd and $\alpha \geq 1$. Then for every divisor $r$ of $n$ with $\frac{n}{r}$ is even and $r \neq \frac{n}{2}$, let $\mathcal{A}_r^e := \{H_i^r \mid i = 1, 2, \dots, \frac{n}{2r}\}$ and $\mathcal{B}_r^e := \{H_i^r \mid i = \frac{n}{2r}+1, \frac{n}{2r}+2, \dots, \frac{n}{r}\}$. For every divisor $r$ of $n$ with $\frac{n}{r}$ is odd and $r \neq n$, let $\mathcal{C}_r^e := \{H_i^r \mid i = 1, 2, \dots, \frac{n}{r}\}$. Here each of $\mathcal{A}_r^e$, $\mathcal{B}_r^e$ and $\mathcal{C}_r^e$ forms a maximal independent set in $\Gamma_N(D_n)$. Also these three class of sets are mutually disjoint and

$$\Gamma_N(D_n) = \bigcup_{\substack{r|n \\ r\neq n \\ \frac{n}{r} \text{ is odd}}} \mathcal{C}_r^e \cup \bigcup_{\substack{r|n \\ r\neq \frac{n}{2} \\ \frac{n}{r} \text{ is even}}} (\mathcal{A}_r^e \cup \mathcal{B}_r^e).$$

The number of divisors $r$ of $n$ such that $r \neq \frac{n}{2}$ with $\frac{n}{r}$ is even is $\alpha\tau(n') - 1$. Each of these divisors gives rise to two partition namely, $A_r^e$ and $B_r^e$ in $V(\Gamma_N(D_n))$. The number of divisors $r$ of $n$ such that $r \neq n$ with $\frac{n}{r}$ is odd is $\tau(n') - 1$. Each of these divisors

gives rise to exactly one partition namely, $C_r^e$ in $V(\Gamma_N(D_n))$. Thus in total, we have $2(\alpha\tau(n') - 1) + \tau(n') - 1 = (2\alpha + 1)\tau(n') - 3$ partitions of $V(\Gamma_N(D_n))$.

**Theorem 4.2.** *Let $n \geq 3$ be an integer. Then $\Gamma_N(D_n)$ is $(\tau(n) - 1)$-partite if $n$ is odd; $((2\alpha + 1)\tau(n') - 3)$-partite if $n = 2^\alpha n'$, with $n'$ is odd, $\alpha \geq 1$. But $\Gamma_N(D_n)$ is not a complete partite graph for any $n$.*

*Proof.* Partiteness of $\Gamma_N(D_n)$ follows from the above discussions. Now we prove the last part of the theorem. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where $p_i$'s are distinct primes and $\alpha_i \geq 1$.
**Case 1.** $n$ is odd. If $k = 1$ and $\alpha_1 = 1$, then by Theorem 4.3, $\Gamma_N(D_n)$ is not complete partite graph. Otherwise, let $H_1^1 \in \mathcal{A}_1$ and $H_2^r \in \mathcal{A}_r$, where $r$ is a divisor of $n$, $r \neq n$. Here $H_1^1$ does not permutes with $H_2^r$ and so $\Gamma_N(D_n)$ is not a complete partite graph.
**Case 2.** $n$ is even. Let $H_1^1 \in \mathcal{A}_1$ and $H^1_{2+\frac{n}{2}} \in \mathcal{A}_{1+\frac{n}{2}}$. Here $H_1^1$ does not permute with $H^1_{2+\frac{n}{2}}$ and so $\Gamma_N(D_n)$ is not a complete partite graph. □

**Theorem 4.3.** *Let $n \geq 3$ be an integer. Then $\Gamma_N(D_n)$ is totally disconnected if and only if $n$ is an odd prime.*

*Proof.* Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where $p_i$'s are distinct primes and $\alpha_i \geq 1$.
**Case 1.** $k = 1$.
**subcase 1.** $n$ is odd. If $\alpha_1 = 1$, then no $H_i^1$ permutes with any $H_j^1$, for every $i, j \in \{1, 2, \dots, p\}$, $i \neq j$ and so $\Gamma_N(D_p) \cong \overline{K}_p$. If $\alpha > 1$, then $H_1^p$, $H_1^1$ permutes. So $\Gamma_N(D_n)$ contains an edge.
**subcase 2.** $n$ is even. Then $H_1^1$ and $H^1_{1+\frac{n}{2}}$ permutes. Therefore $\Gamma_N(D_n)$ contains an edge.
**Case 2.** $k \geq 2$. If $n$ is odd, then $H_1^{p^{\alpha_1}}$ and $H_1^{p^{\alpha_2}}$ permutes and so $\Gamma_N(D_n)$ contains an edge. If $n$ is even, then $H_1^1$, $H^1_{1+\frac{n}{2}}$ permutes and so $\Gamma_N(D_n)$ contains an edge. Combaining all the cases together gives the result. □

**Theorem 4.4.** *Let $n \geq 3$ be an integer.*

(i) $$\alpha(\Gamma_N(D_n)) = \begin{cases} n, & \text{if } n \text{ is odd;} \\ \frac{n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

(ii) $\gamma(\Gamma_N(D_n)) = p$, *where* $p$ *is the smallest prime factor of* $n$.

(iii) $\omega(\Gamma_N(D_n)) = \chi(\Gamma_N(D_n)) = \begin{cases} \tau(n) - 1, & \textit{if } n \textit{ is odd;} \\ (2\alpha + 1)\tau(n') - 3, & \textit{if } n = 2^\alpha n', \textit{with } n' \textit{ is odd.} \end{cases}$

*In particular,* $\Gamma_N(D_n)$ *is weakly perfect.*

*Proof.* (i): As explained earlier, $V(\Gamma_N(D_n))$ is the disjoint union of maximal independent sets. If $n$ is odd, the maximal independent set $\mathcal{A}_1^o$ in $\Gamma_N(D_n)$ has the maximal cardinality with $|\mathcal{A}_1^o| = n$. If $n$ is even, the maximal independent sets $\mathcal{A}_1^e$ and $\mathcal{B}_1^e$ in $\Gamma_N(D_n)$ both attains the maximal cardinality with $|\mathcal{A}_1^e| = \frac{n}{2} = |\mathcal{B}_1^e|$.
(ii): Note that $\omega(\Gamma_N(D_n)) \leq \chi(\Gamma_N(D_n))$, for every $n$.
**Case 1.** $n$ is odd. Then $\mathcal{A} := \{H_1^r \mid r|n,\ r \neq n\}$ is a clique set in $\Gamma_N(D_n)$. So $\omega(\Gamma_N(D_n)) \geq |\mathcal{A}| = \tau(n) - 1$. By Theorem 4.2, $\Gamma_N(D_n)$ is a $\tau(n) - 1$ partite graph. So $\chi(\Gamma_N(D_n)) \leq \tau(n) - 1$. Thus $\tau(n) - 1 \leq \omega(\Gamma_N(D_n)) \leq \chi(\Gamma_N(D_n)) \leq \tau(n) - 1$. Therefore, $\omega(\Gamma_N(D_n)) = \chi(\Gamma_N(D_n)) = \tau(n) - 1$.
**Case 2.** $n = 2^\alpha p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where $p_i$'s are distinct primes and $\alpha$, $\alpha_i \geq 1$.

Then Let $\mathcal{A} := \{H_1^r, H_{1+\frac{n}{2r}}^r \mid r \text{ is a divisor of } n \text{ with } \frac{n}{r} \text{ is even}, r \neq n, 2^{\alpha-1}p_2^{\alpha_2} \dots p_k^{\alpha_k}\} \cup \{H_1^r \mid r \text{ is a divisor of } n \text{ with } \frac{n}{r} \text{ is odd and } r \neq n\}$ forms a clique set in $\Gamma_N(D_n)$.

In this case, $\omega(\Gamma_N(D_n)) \geq |\mathcal{A}| = (2\alpha + 1)\tau(n') - 3$. By Theorem 4.2, $\Gamma_N(D_n)$ is a $(2\alpha + 1)\tau(n') - 3$ partite graph and so $\chi(\Gamma_N(D_n)) \leq (2\alpha + 1)\tau(n') - 3$. Thus $(2\alpha+1)\tau(n')-3 \leq \omega(\Gamma_N(D_n)) \leq \chi(\Gamma_N(D_n)) \leq (2\alpha+1)\tau(n')-3$. Therefore, $\omega(\Gamma_N(D_n)) = \chi(\Gamma_N(D_n)) = (2\alpha + 1)\tau(n') - 3$.

Weakly perfectness of $\Gamma_N(D_n)$ follows by the above two cases.
(iii): By Theorem 4.2, $\Gamma_N(D_n)$ is a partite graph and every partition of $V(\Gamma_N(D_n))$ is an maximal independent set. Among these maximal independent sets, $\mathcal{A}_d$ is a maximal independent set in $\Gamma_N(D_n)$ with minimum cardinality, where $d$ is the largest divisor of $n$. It is well known that in a graph any maximal independent set is a dominating set. Consequently, $\mathcal{A}_d$ is an independent dominating set of $\Gamma_N(D_n)$ with minimum cardinality, where $d$ is the largest divisor of $n$.

If $n = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ is even, where $p_i$'s are distinct primes and $\alpha$, $\alpha_i \geq 1$, then

$d = 2^{\alpha-1}p_2^{\alpha_2}\ldots, p_k^{\alpha_k}$ is the largest divisor of $n$ and so $|\mathcal{A}_d| = 2$. Hence $\gamma(\Gamma_N(D_n)) \leq 2$. Suppose $\gamma(\Gamma_N(D_n)) = 1$, then there exists a subgroup in $D_n$, which permutes with every other subgroups of $D_n$, which is not possible, since by Theorem 4.2, $\Gamma_N(D_n)$ is a partite graph with every partition has at least two vertices. So $\gamma(\Gamma_N(D_n)) = 2$.

Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$ be odd, where $p_i$'s are distinct primes and $\alpha_i \geq 1$. With out loss of generality we assume that $p_1 < p_2 < \ldots < p_k$. Then $d = p_1^{\alpha_1-1}p_2^{\alpha_2}\ldots p_k^{\alpha_k}$ is the largest divisor of $n$ and so $|\mathcal{A}_d| = p_1$. Thus $\gamma(\Gamma_N(D_n)) \leq p_1$. We show that, $\mathcal{A}_d$ is a dominating set with minimal cardinality. Now for any $i \in \{1, 2, \ldots, \frac{n}{d}\}$, we consider $\mathcal{A}_d - \{H_i^d\}$. Then no vertex in $\mathcal{A}_d - \{H_i^d\}$ permutes with $H_i^1$. It follows that $p_1 - 1 < \gamma(\Gamma_N(D_n)) \leq p_1$. Therefore, $\gamma(\Gamma_N(D_n)) = p_1$. $\square$

In the next result, we give the structure of $\Gamma_N(D_n)$ for some values of $n$.

**Theorem 4.5.** *Let $n \geq 3$ be an integer.*

(i) *If $n = 2^\alpha$, $\alpha \geq 2$, then*

$$\Gamma_N(D_n) \cong 2(\Gamma_N(D_{\frac{n}{2}}) + K_2) \tag{4.16}$$

*with $\Gamma_N(D_{2^2}) \cong 2K_2$.*

(ii) *If $n = p^\alpha$, where $p$ is a prime, $p > 2$, $\alpha \geq 1$, then*

$$\Gamma_N(D_n) \cong p(\Gamma_N(D_{\frac{n}{p}}) + K_1) \tag{4.17}$$

*with $\Gamma_N(D_p) \cong \overline{K}_p$.*

(iii) *If $n = 2p$, where $p$ is a prime, $p > 2$, then*

$$\Gamma_N(D_n) \cong pK_3. \tag{4.18}$$

(iv) *If $n = pq$, where $p, q$ are distinct primes, $2 < p < q$, then*

$$\Gamma_N(D_n) \cong \bigcup_{i=1}^{p}\bigcup_{j=1}^{q} \mathcal{G}_{ij}, \tag{4.19}$$

*where for each $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$, $\mathcal{G}_{ij}$ is the complete graph with vertex set $\{u_i, v_j, w_{ij}\}$.*

*Proof.* (i): If $\alpha = 2$, then $H_i^1$, $i = 1, 2, 3, 4$ are the only non-normal subgroups of $D_n$. Also $H_i^1$ and $H_{i+2}^1$ permutes with each other for each $i \in \{1, 2\}$; no two remaining subgroups permutes. It follows that, $\Gamma_N(D_n) \cong 2K_2$. Now we consider $\alpha > 2$. For each $i = 1, 2$, let $\mathcal{G}_i$ denotes the subgraph of $\Gamma_N(D_n)$ induced by the proper non-normal subgroups in $H_i^{2^{\alpha-1}}$. Clearly $\Gamma_N(D_n) \cong \mathcal{G}_1 \cup \mathcal{G}_2$. Since for each $i = 1, 2$ $H_i^{2^{\alpha-1}} \cong D_{2^{\alpha-1}}$, so we have

$$\Gamma_N(H_i^{2^{\alpha-1}}) \cong \Gamma_N(D_{2^{\alpha-1}}). \tag{4.20}$$

For each $i = 1, 2$, $H_i^{2^{\alpha-2}}$ and $H_{i+2}^{2^{\alpha-2}}$ permutes in $D_n$ and they are normal in $H_i^{2^{\alpha-1}}$; but are not normal in $D_n$. So by using (4.20), for each $i = 1, 2$ we have

$$\mathcal{G}_i \cong \Gamma_N(H_i^{2^{\alpha-1}}) + K_2 \cong \Gamma_N(D_{2^{\alpha-1}}) + K_2.$$

Here no vertex of $\mathcal{G}_1$ is adjacent with any vertex of $\mathcal{G}_2$. Thus $\Gamma_N(D_n)$ is the disjoint union of $\mathcal{G}_1$ and $\mathcal{G}_2$.

(ii): If $\alpha = 1$, then $H_i^1$, $i = 1, 2, \ldots, p$ are the only non-normal subgroups of $D_n$. No two distinct subgroups of this type permutes. Therefore, $\Gamma_N(D_n) \cong \overline{K}_p$. Now we consider $\alpha > 1$. For each $i = 1, 2, \ldots, p$, let $\mathcal{G}_i$ denotes the subgraph of $\Gamma_N(D_n)$ induced by the proper non-normal subgroups in $H_i^{p^{\alpha-1}}$. Clearly $\Gamma_N(D_n) \cong \bigcup_{i=1}^{p} \mathcal{G}_i$. Since for each $i = 1, 2, \ldots, p$, $H_i^{p^{\alpha-1}} \cong D_{p^{\alpha-1}}$, so we have

$$\Gamma_N(H_i^{p^{\alpha-1}}) \cong \Gamma_N(D_{p^{\alpha-1}}). \tag{4.21}$$

Here $H_i^{p^{\alpha-1}}$, $i = 1, 2, \ldots, p$ permutes with all its proper subgroups. Further no $H_i^{\alpha-1}$

permutes with any $H_j^{\alpha-1}$ and its subgroups for all $i,\ j = 1, 2, \dots, p,\ i \neq j$. So by using (4.21), for each $i = 1, 2, \dots, p$, we have

$$\mathcal{G}_i \cong \Gamma_N(H_i^{p^{\alpha-1}}) + K_1 \cong \Gamma_N(D_{p^{\alpha-1}}) + K_1.$$

So $\Gamma_N(D_n)$ is the disjoint union of $\mathcal{G}_i$'s.

(iii): If $n = 2p$, then $H_i^2$, $i = 1, \dots, p$ and $H_i^1$, $i = 1, 2, \dots, 2p$ are the only non-normal subgroups of $D_n$. Also for each $i \in \{1, \dots, p\}$, $H_i^1$, $H_{i+p}^1$ are subgroups of $H_i^2$; they permutes with each other; no $H_i^2$ permutes with $H_j^2$ for every $i \neq j$. It follows that $\Gamma_N(D_n) \cong pK_3$.

(iv): If $n = pq$, $2 < p < q$, then $H_i^q$, $i = 1, \dots, p$, $H_j^p$, $j = 1, \dots, q$ and $H_k^1$, $k = 1, \dots, pq$ are the only non-normal subgroups of $D_n$. Here for each $i \in \{1, \dots, p\}$, $j \in \{1, \dots, q\}$, $H_i^q$ permutes with $H_j^p$; no two subgroups of the form $H_i^q$ permutes with each other. For each $i \in \{1, \dots, p\}$, $j \in \{1, \dots, q\}$, by Chinese Remainder Theorem, there exist a positive integer, say $m_{ij}$ such that $m_{ij} \equiv i \mod p$ and $m_{ij} \equiv j \mod q$. It follows that, $H_{m_{ij}}^1$ is the unique proper subgroup of $H_i^q$ and $H_j^p$ for every $i = 1, 2, \dots, p$, $j = 1, 2, \dots, q$. So for each $i \in \{1, \dots, p\}$, $j \in \{1, \dots, q\}$, $H_i^q$, $H_j^p$, $H_{m_{ij}}^1$ permutes with each other and hence they form the complete graph, say $\mathcal{G}_{ij}$ as a subgraph of $\Gamma_N(D_n)$ with vertex set $\{H_i^q, H_j^p, H_{m_{ij}}^1\}$. Thus $\Gamma_N(D_n)$ is the union of these $\mathcal{G}_{ij}$'s. $\square$

**Example 4.1.** *In Figure 1, we exhibit the structure of* $\Gamma_N(D_{15})$ *described in Theorem 4.5*(iv).

**Theorem 4.6.** *If* $n \geq 3$ *is a deficient number, then* $\Gamma_N(D_n)$ *is non-Hamiltonian.*

*Proof.* Let $S = V(\Gamma_N(D_n)) - \mathcal{A}_1$. Since $n$ is an deficient number, we have $|S| = \sigma(n) - (n + 1) < n = |\mathcal{A}_1| = c(\Gamma_N(D_n) - S)$ and so by Theorem 2.1, $\Gamma_N(D_n)$ is non-Hamiltonian. $\square$

**Corollary 4.3.** *Let* $p$, $q$ *be two distinct primes. If* $n$ *is one of the following:* $n = 2^\alpha$, $\alpha \geq 2$, $n = p^\alpha$, $\alpha \geq 1$, $n = 2p$, $p > 2$, *or* $n = pq$, $2 < p < q$, *then* $\Gamma_N(D_n)$ *is non-Hamiltonian.*

*Proof.* If $n$ is one of the following: $n = 2^\alpha$, $\alpha \geq 2$, $n = p^\alpha$, $\alpha \geq 1$, or $n = 2p$, $p > 2$,

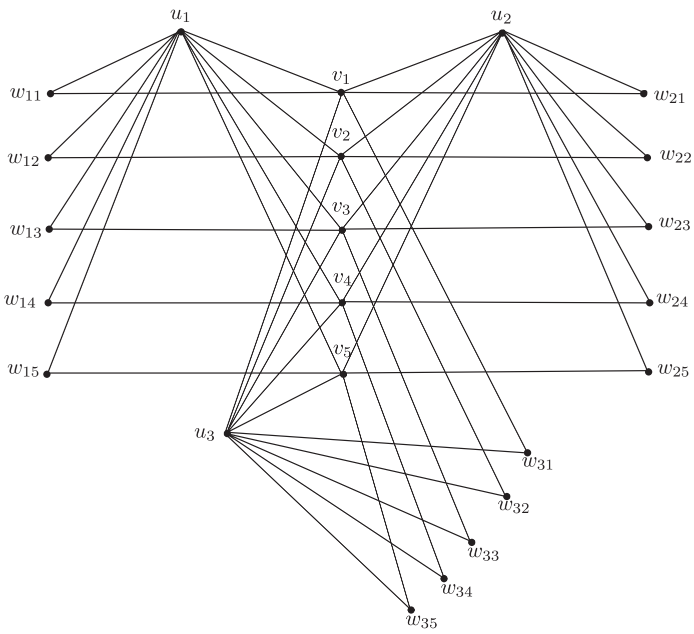


Figure 1: The graph $\Gamma_N(D_{15})$

then by (4.16), (4.17), and (4.18), respectively $\Gamma_N(D_n)$ is disconnected and hence $\Gamma(D_n)$ is non-Hamiltonian.

If $n = pq$, then $n$ is a deficient number and so by Theorem 4.6, $\Gamma_N(D_n)$ is non-Hamiltonian. □

**Theorem 4.7.** *If $n = 2^\alpha$, $\alpha \geq 2$, then each component of $\Gamma_N(D_n)$ contains a Hamiltonian path.*

*Proof.* We shall prove this result by induction on $\alpha$. If $\alpha = 2$, then by Theorem 4.5(i), $\Gamma_N(D_n) \cong 2K_2$. Here each component contains a Hamiltonian path. Hence the result is true for $\alpha = 2$. We assume that the result is true for any positive integer $m < \alpha$. Now we consider $\alpha > 2$. By Therorem 4.5(i), $\Gamma_N(D_n)$ has two components, say $G_1$ and $G_2$ each isomorphic to $\Gamma_N(D_{2^{\alpha-1}}) + K_2$. Let us consider any one of these components, say $G_1$. By induction hypothesis and by Therorem 4.5(i), $\Gamma_N(D_{2^{\alpha-1}})$ has two components and each contains a Hamiltonian path, say $P$ and $P'$ respectively. Also let $u$ and $v$ be the vertices of $K_2$. Then we have a Hamiltonian path, $u - P - v - P'$ in $G_1$. This completes the proof. □

## 4.2 Properties of $\Gamma(D_n)$

Now we start to investigate the structure and properties of $\Gamma(D_n)$.

**Theorem 4.8.** *Let $n \geq 3$ be an integer. Then $\Gamma(D_n) \cong K_r + \Gamma_N(D_n)$, where $r = \tau(n)+1$ if $n$ is even; $r = \tau(n) - 1$ if $n$ is odd.*

*Proof.* Since the number of normal subgroups of $D_n$ is $\tau(n) + 1$ if $n$ is even; $\tau(n) - 1$ if $n$ is odd, so the proof follows by Theorems 3.1 and 4.2. ☐

**Corollary 4.4.** *Let $n \geq 3$ be an integer.*

(i) $\deg_{\Gamma(D_n)}(H_o^r) = \tau(n) + \sigma(n) - 3$, *for every divisor $r$ of $n$, $r \neq 1$.*

(ii) $\deg_{\Gamma(D_n)}(H_i^r) = \tau(n) + x_i^r - 3$, *for every divisor $r$ of $n$, $r \neq n$, $i = 1, 2, \ldots, \frac{n}{r}$, where $x_i^r$ is given by* (4.4) *if $n$ is odd; $x_i^r$ is given by* (4.8) *if $n = 2^\alpha$, $\alpha \geq 2$; $x_i^r$ is given by* (4.10) *if $n = 2^\alpha n'$, with $n'$ is odd and $\alpha \geq 1$.*

*Proof.* (i): For each divisor $r$ of $n$, $r \neq 1$, the subgroup $H_0^r$ is normal in $D_n$ for any $n$. So by Corollary 3.1(i) and (4.1), $\deg_{\Gamma(D_n)}(H_0^r) = \tau(n) + \sigma(n) - 3$.
(ii): Follows by Theorem 4.1, Corollary 3.1(i) and (4.1). ☐

**Corollary 4.5.** *Let $n \geq 3$ be an integer and $g$ denotes the arithmetic function given by* (4.7).

(i) *If $n$ is odd, then*

$$|E(\Gamma(D_n))| = \frac{1}{2}\big\{g(n) - 5\sigma(n) + \tau(n)[\tau(n) + 2\sigma(n) - 5] + 6\big\}.$$

(ii) *If $n = 2^\alpha$, $\alpha \geq 2$, then*

$$|E(\Gamma(D_n))| = \frac{1}{2}\big\{2^{\alpha+1}(6\alpha - 7) + \alpha^2 - 5\alpha + 14\big\}.$$

(iii) *If $n = 2^\alpha n'$, with $n'$ is odd, $\alpha \geq 1$, then*

$$|E(\Gamma(D_n))| = \frac{1}{2}\big\{[(\alpha - 1)2^{\alpha+3} + 9]g(n') - 5\sigma(n) + \tau(n) + 6 + \tau(n)[\tau(n) + 2\sigma(n) - 6]\big\}.$$

*Proof.* Follows by Corollaries 4.1, 3.1(ii), (4.1) and from the fact that the number of normal subgroups of $D_n$ is $\tau(n)+1$ if $n$ is even; $\tau(n)-1$ if $n$ is odd. □

**Corollary 4.6.** *Let $n \geq 3$ be an integer.*

(i) *If $n$ is odd, then $\Gamma(D_n)$ is non-Eulerian.*

(ii) *If $n = 2^{\alpha}$, $\alpha \geq 2$, then $\Gamma(D_n)$ is Eulerian if and only if $\alpha$ is odd.*

(iii) *If $n = 2^{\alpha} n'$, with $n' = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$ is odd, where $p_i$'s are distinct primes and $\alpha, \alpha_i \geq 1$, then $\Gamma(D_n)$ is Eulerian if and only if $\alpha$ is odd and $\alpha_i$ is even for every $i \in \{1, 2, \dots, k\}$.*

*Proof.* (i): By Corollary 4.4(ii) and (4.13), $\deg_{\Gamma(D_n)}(H_i^1)$ is odd, for each $i = 1, 2, \dots, n$. So $\Gamma(D_n)$ is non-Eulerian.
(ii): By Corollary 4.4(i), $\deg_{\Gamma(D_n)}(H_0^r)$ is even if and only if either $\tau(n)$ is odd and $\sigma(n)$ is even or $\tau(n)$ is even and $\sigma(n)$ is odd. By Note 2.1, it follows that $\deg_{\Gamma(D_n)}(H_0^r)$ is even if and only if $\alpha$ is odd. Also by Corollary 4.4(ii), $\deg(H_i^r)$ is even if and only if either $\tau(n)$ is odd and $x_i^r$ is even or $\tau(n)$ is even and $x_i^r$ is odd. By (4.8) $x_i^r$ is odd. Further $\tau(n)$ is even only when $\alpha$ is odd. Thus $\deg(H_i^r)$ is even only when $\alpha$ is odd. Combining all these, we get the result.
(iii): By Corollary 4.4(i), $\deg_{\Gamma(D_N)}(H_0^r)$ is even if and only if either $\tau(n)$ is odd and $\sigma(n)$ is even or $\tau(n)$ is even and $\sigma(n)$ is odd. By Note 2.1, we have $\tau(n)$ is even and $\sigma(n)$ is odd. It follows that $\deg_{\Gamma(D_n)}(H_0^r)$ is even if and only if $\alpha$ is odd and $\alpha_i$ is even for every $i \in \{1, 2, \dots, k\}$. Also by Corollary 4.4(ii), $\deg_{\Gamma(D_n)}(H_i^r)$ is even if and only if either $\tau(n)$ is odd and $x_i^r$ is even or $\tau(n)$ is even and $x_i^r$ is odd. By the above argument, we have $\tau(n)$ is even and so we must have $x_i^r$ is odd. By (4.10), $x_i^r$ is odd if and only if $\sigma(n')$ is odd. Now, by Note 2.1, it follows that $\deg_{\Gamma(D_n)}(H_i^r)$ is even if and only if $\alpha_i$ is even for every $i \in \{1, 2, \dots, k\}$. Proof follows by combining the above facts. □

**Corollary 4.7.** *Let $n \geq 3$ be an integer.*

(i) $\alpha(\Gamma(D_n)) = \begin{cases} n, & \textit{if } n \textit{ is odd;} \\ \frac{n}{2}, & \textit{if } n \textit{ is even} \end{cases}$

(ii) $\omega(\Gamma(D_n)) = \chi(\Gamma(D_n)) = \begin{cases} 2(\tau(n)-1), & \text{if } n \text{ is odd;} \\ \tau(n)+(2\alpha+1)\tau(n')-2, & \text{if } n = 2^\alpha n', \text{with } n' \text{ is odd.} \end{cases}$

*In particular,* $\Gamma(D_n)$ *is weakly perfect.*

*Proof.* (i): Proof follows by Corollary 3.1 and Theorem 4.4.

(ii): Since the number of proper normal subgroups of $D_n$ is $\tau(n)-1$ if $n$ is odd; $\tau(n)+1$ if $n$ is even, so the proof follows by Theorem 4.4 and Corollary 3.1. □

**Theorem 4.9.** *Let* $n \geq 3$ *be an integer. Then* $\Gamma(D_n)$ *is a split graph if and only if* $n$ *is an odd prime.*

*Proof.* By Theorem 4.2, $\Gamma_N(D_n)$ is a partite graph, so by definition of split graph and by Theorem 4.3, $\Gamma(D_n)$ is a split graph if and only if $n$ is an odd prime. □

In the following result, we describe the structure of $\Gamma(D_n)$ for some values of $n$.

**Corollary 4.8.** *Let* $n \geq 3$ *be an integer.*

(i) *If* $n = 2^\alpha$ *and* $\alpha \geq 2$*, then* $\Gamma(D_n) \cong K_{\alpha+2} + \Gamma_N(D_n)$*, where* $\Gamma_N(D_n)$ *is given by* (4.16).

(ii) *If* $n = p^\alpha$ *where* $p$ *is a prime,* $p > 2$*,* $\alpha \geq 1$*, then* $\Gamma(D_n) \cong K_\alpha + \Gamma_N(D_n)$*, where* $\Gamma_N(D_n)$ *is given by* (4.17).

(iii) *If* $n = 2p$*, where* $p$ *is a prime,* $p > 2$*, then* $\Gamma(D_n) \cong K_5 + \Gamma_N(D_n)$*, where* $\Gamma_N(D_n)$ *is given by* (4.18).

(iv) *If* $n = pq$*, where* $p$*,* $q$ *are distinct primes,* $2 < p < q$*, then* $\Gamma(D_n) \cong K_3 + \Gamma_N(D_n)$*, where* $\Gamma_N(D_n)$ *is given by* (4.19).

*Proof.* Follows from Theorems 4.8 and 4.5. □

**Theorem 4.10.** *Let* $n \geq 3$ *be an integer. If* $\tau(n)+\sigma(n) < 2(n+1)$*, then* $\Gamma(D_n)$ *is non-Hamiltonian.*

*Proof.* Let $S = V(\Gamma(D_n)) - \mathcal{A}_1$. Since $\tau(n)+\sigma(n) < 2(n+1)$, so $|S| = \tau(n)+\sigma(n)-(n+2) < n = |\mathcal{A}_1| = c(\Gamma(D_n) - S)$. Thus by Theorem 2.1, $\Gamma(D_n)$ is non-Hamiltonian. □

**Lemma 4.1.** *Let $G_r$, $r = 1, 2, \ldots, m$ be vertex disjoint graphs and $H$ be a graph with $n$ vertices. Suppose that $G_r$, $r = 1, 2, \ldots, m$ and $H$ have Hamiltonian paths. Let $G \cong H + \bigcup_{r=1}^{m} G_r$. Then $G$ is Hamiltonian if and only if $m \leq n$.*

*Proof.* For each $r = 1, 2, \ldots, m$, let $P_r$ be a Hamiltonian path in $G_r$ and let $P : v_1 - v_2 - \cdots - v_n$ be a Hamiltonian path in $H$. If $m \leq n$, then $v_1 - P_2 - v_2 - P_3 - v_3 - \cdots - v_{m-1} - P_m - v_m - v_{m+1} \cdots - v_n - P_1 - v_1$ is a Hamiltonian cycle in $G$. If $m > n$, we take $S = V(H)$. Then $c(G - S) > |S|$ and so by Theorem 2.1, $G$ is non-Hamiltonian. $\square$

**Theorem 4.11.**

(i) *If $n = p^\alpha$, where $p$ is prime, $\alpha \geq 1$, then $\Gamma(D_n)$ is Hamiltonian if and only if $p = 2$ and $\alpha \geq 2$.*

(ii) *If $n = pq$, where $p$, $q$ are distinct primes, $p < q$, then $\Gamma(D_n)$ is Hamiltonian if and only if $p = 2$ and $q \leq 5$.*

*Proof.* (i): We need to consider the following cases:

**Case a.** $p = 2$. By Corollary 4.8(i), $\Gamma(D_n) \cong H + \bigcup_{r=1}^{2} G_r$, where $H \cong K_{\alpha+2}$ and $G_r \cong \Gamma(D_{\frac{n}{2}}) + K_2$. Now by Theorem 4.7 and Lemma 4.1, $\Gamma(D_n)$ is Hamiltonian.

**Case b.** $p > 2$. Let $S = V(\Gamma(D_n)) - \mathcal{A}_1$. Then $|S| = \tau(n) + \sigma(n) - (n+2) < n = |\mathcal{A}_1| = c(\Gamma(D_n) - S)$. So by Theorem 2.1, $\Gamma(D_n)$ is non-Hamiltonian.

(ii): We give the proof in the following cases:

**Case a.** $p = 2$. Then by Corollary 4.8(iii), $\Gamma(D_n) \cong H + \bigcup_{r=1}^{q} G_r$, where $H \cong K_5$ and for each $r = 1, 2, \ldots, q$, $G_r \cong K_3$. Then by Lemma 4.1, $\Gamma(D_n)$ is Hamiltonian if and only if $q \leq 5$.

**Case b.** $p \geq 3$ and $q \geq 5$. Let $S = V(\Gamma(D_n)) - \mathcal{A}_1$. Then $|S| = \tau(n) + \sigma(n) - (n+4) < n = |\mathcal{A}_1| = c(\Gamma(D_n) - S)$. So by Theorem 2.1, $\Gamma(D_n)$ is non-Hamiltonian. $\square$

# 5 Quaternion groups

For any integer $n > 1$, the quaternion group of order $4n$, is defined by

$$Q_n = \langle a, b | a^{2n} = b^4 = 1, b^2 = a^n, ab = ba^{-1} \rangle.$$

The subgroups of $Q_n$ are listed below:

(i) cyclic groups $H_{0,r} = \langle a^{\frac{2n}{r}} \rangle$, of order $r$, where $r$ is a divisor of $2n$;

(ii) cyclic groups $H_{i,1} = \langle a^i b \rangle$ of order 4, where $i = 1, \ldots, n$;

(iii) quaternion groups $H_{i,r} = \langle a^{\frac{n}{r}}, a^i b \rangle$ of order $4r$, where $r$ is a divisor of $n$, $i = 1, \ldots, \frac{n}{r}$.

The proper normal subgroups of $Q_n$ are the subgroups $H_0^r$, $r \neq 1$ listed in (i), when $n$ is odd; the subgroups $H_0^r$, $r \neq 1$ listed in (i) and $H_{i,\frac{n}{2}}$, $i = 1, 2$ of index 2, when $n$ is even. Thus

$$|L(Q_n)| = \tau(2n) + \sigma(n), \tag{5.1}$$

and so

$$|V(\Gamma(Q_n))| = \tau(2n) + \sigma(n) - 2,$$

$$|V(\Gamma_N(D_n)| = \begin{cases} \sigma(n) - 1, & \text{if } n \text{ is odd;} \\ \sigma(n) - 3, & \text{otherwise.} \end{cases}$$

It is well known that $Z(Q_n) \cong \langle a^n \rangle$ is the unique minimal subgroup of $Q_n$ and

$$\frac{Q_n}{Z(Q_n)} \cong D_n.$$

For each positive divisors $r, s$ of $n$ and $i \in \{1, 2 \ldots, \frac{n}{r}\}$, $j \in \{1, 2 \ldots, \frac{n}{s}\}$, consider the subgroups $H_{i,r}$ and $H_{j,s}$, of $Q_n$.

$H_{i,r}$ and $H_{j,s}$ permutes if and only if $\dfrac{H_{i,r}}{Z(Q_n)}$ and $\dfrac{H_{j,s}}{Z(Q_n)}$ permutes.

But

$$\frac{H_{i,r}}{Z(Q_n)} \cong \langle x^{\frac{n}{r}}, yx^{i'-1} \rangle \cong H_{i'}^r$$

and

$$\frac{H_{j,s}}{Z(Q_n)} \cong \langle x^{\frac{n}{s}}, yx^{j'-1} \rangle \cong H_{j'}^s,$$

where $x = a\langle a^n \rangle$, $y = b\langle a^n \rangle$, $i = q_1 r + i'$, $j = q_2 s + j'$, $0 \leq i' < r$ and $0 \leq j' < s$ for some $q_1$, $q_2 \in \mathbb{Z}$.

Thus $H_{i,r}$ and $H_{j,s}$ permutes if and only if $H_i^r$ and $H_j^s$ permutes.

In view of these together with the necessary and sufficient condition for the permutability of dihedral subgroups discussed in Section 4, we have the following:

$H_{i,r}$ permutes with $H_{j,s}$ if and only if $x^{2(i'-j')} \in \langle x^{\frac{n}{[r,s]}} \rangle$, that is,

$$\frac{n}{[r,s]} \;\Big|\; 2(i'-j'). \tag{5.2}$$

Let $r$ be a fixed positive divisor of $n$ and $i \in \{1, 2, \ldots, \frac{n}{r}\}$. Suppose that $i = q'r + i'$, where $0 \leq i' < r$, $q' \in \mathbb{Z}$. Then it is easy to see that $x_{i'}^r$, the number of solution of (5.2) is equal to $x_i^r$, the number of solution of (4.3).

In view of these facts, we have the following result.

**Theorem 5.1.** *Let $n \geq 3$ be an integer. Then $\Gamma_N(Q_n) \cong \Gamma_N(D_n)$.*

**Theorem 5.2.** *Let $n > 1$ be an integer. Then $\Gamma(Q_n) \cong K_r + \Gamma_N(D_n)$, where $r = \tau(2n)+1$ if $n$ is even; $r = \tau(2n) - 1$ if $n$ is odd.*

*Proof.* Since the number of normal subgroups of $Q_n$ is $\tau(2n) + 1$ if $n$ is even; $\tau(2n) - 1$ if $n$ is odd, so the proof follows by Theorems 3.1, 5.1 and 4.2. $\square$

**Corollary 5.1.** *Let $n > 1$ be an integer.*

(i) $\deg_{\Gamma(Q_n)}(H_{0,r}) = \tau(2n) + \sigma(n) - 3$, *for every divisor $r$ of $2n$, $r \neq 2n$.*

(ii) $\deg_{\Gamma(Q_n)}(H_{i,r}) = \tau(2n) + {x_i}^r - 3$, *for every divisor $r$ of $n$, $r \neq n$, $i \in \{1, 2, \ldots, \frac{n}{r}\}$, where $x_i^r$ is given by (4.4) if $n$ is odd; where $x_i^r$ is given by (4.8) if $n = 2^\alpha$, $\alpha \geq 1$; $x_i^r$ is given by (4.10) if $n = 2^\alpha n'$, with $n'$ is odd and $\alpha \geq 1$.*

*Proof.* Follows by Theorems 5.1, 4.1, Corollary 3.1 and (5.1). $\square$

**Corollary 5.2.** *Let $n > 1$ be an integer and $g$ denotes the arithmetic function given by (4.7).*

(i) *If $n$ is odd, then*

$$|E(\Gamma(Q_n))| = \frac{1}{2}\big\{g(n) - 5\sigma(n) + 6 + \tau(2n)(\tau(2n) + 2\sigma(n) - 5)\big\}.$$

(ii) *If* $n = 2^\alpha$, $\alpha \geq 2$, *then*

$$|E(\Gamma(Q_n))| = \frac{1}{2}\big\{2^{\alpha+1}(6\alpha - 5) + \alpha^2 - 3\alpha + 10\big\}.$$

(iii) *If* $n = 2^\alpha n'$, *with* $n'$ *is odd,* $\alpha \geq 1$, *then*

$$|E(\Gamma(Q_n))| = \frac{1}{2}\big\{[(\alpha-1)2^{\alpha+3}+9]g(n')-5\sigma(n)+\tau(2n)+6+\tau(2n)[\tau(2n)+2\sigma(n)-6]\big\}.$$

*Proof.* Follows by Theorem 5.1, Corollary 4.1, (5.1) and from the fact that the number of normal subgroups of $Q_n$ is $\tau(2n) + 1$ if $n$ is even; $\tau(2n) - 1$ if $n$ is odd. □

Now we characterize the values of $n$ for which $\Gamma(Q_n)$ is Eulerian.

**Corollary 5.3.** *Let* $n > 1$ *be an integer.*

(i) *If* $n$ *is odd, then* $\Gamma(Q_n)$ *is non-Eulerian.*

(ii) *If* $n = 2^\alpha$, $\alpha \geq 2$, *then* $\Gamma(Q_n)$ *is Eulerian if and only if* $\alpha$ *is even.*

(iii) *If* $n = 2^\alpha n'$, *with* $n'$ *is odd,* $n' = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, *where* $p_i$*'s are distinct primes and* $\alpha$, $\alpha_i \geq 1$, *then* $\Gamma(Q_n)$ *is Eulerian if and only if* $\alpha$ *is even and* $\alpha_i$ *is even, for every* $i \in \{1, 2, \ldots, k\}$.

*Proof.* (i): By Corollary 5.1(ii), $\deg_{\Gamma(Q_n)}(H_{i,1})$ is odd. So proof follows.
(ii): By Corollary 5.1(i), $\deg_{\Gamma(Q_n)}(H_{0,r})$ is even if and only if either $\tau(2n)$ is odd and $\sigma(n)$ is even or $\tau(2n)$ is even and $\sigma(n)$ is odd. By Note 2.1, $\sigma(n)$ is odd, so we must have $\tau(2n)$ is even. By Note 2.1, $\tau(2n)$ is even if and only if $\alpha$ is even. Now by Corollary 5.1(ii), $\deg_{\Gamma(Q_n)}(H_{i,r})$ is even if and only if $x_i^r$ is odd. By the above argument $\tau(2n)$ is even and by (4.8), $x_i^r$ is odd. So the proof follows.
(iii). By Corollary 5.1(i), $\deg_{\Gamma(Q_n)}(H_{0,r})$ is even if and only if either $\tau(2n)$ is even and $\sigma(n)$ is odd or $\tau(2n)$ is odd and $\sigma(n)$ is even. Now we haver to consider following two cases.

**Case a.** $\alpha$ is odd. By Note 2.1, there is no such $n$ exist.

**Case b.** $\alpha$ is even. By Note 2.1, $\tau(2n)$ is even, so we must have $\sigma(n)$ is odd. But $\sigma(n)$ is odd if and only if $\sigma(n')$ is odd, by Note 2.1, $\sigma(n')$ is odd if and only if $\alpha_i$ is even for every $i \in \{1, 2, \ldots, k\}$. Now by Corollary 5.1(ii), $\deg_{\Gamma(Q_n)}(H_{i,r})$ is even if and only if either $\tau(2n)$ is even and $x_i^r$ is odd or $\tau(2n)$ is odd and $x_i^r$ is even. But by the above argument $\tau(2n)$ is even, so we must have $x_i^r$ is odd. By (4.10) $x_i^r$ is odd if and only if $\sigma(n')$ is odd. Thus by Note 2.1, $x_i^r$ is odd if and only if $\alpha_i$ is even for every $i \in \{1, 2, \ldots, k\}$. □

**Corollary 5.4.** *Let $n > 1$ be an integer.*

(i) $$\alpha(\Gamma(Q_n)) = \begin{cases} n, & \text{if } n \text{ is odd;} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

(ii) $$\omega(\Gamma(Q_n)) = \chi(\Gamma(Q_n)) = \begin{cases} \tau(2n) + \tau(n) - 2, & \text{if } n \text{ is odd;} \\ \tau(2n) + (2\alpha + 1)\tau(n') - 2, & \text{if } n = 2^\alpha n', \text{ with } n' \text{ is odd.} \end{cases}$$

*In particular, $\Gamma(Q_n)$ is weakly perfect.*

*Proof.* (i): Proof follows by Theorems 5.1, 4.4 and Corollary 3.1.

(ii): Since the number of proper normal subgroups of $Q_n$ is $\tau(2n) - 1$ if $n$ is odd; $\tau(2n) + 1$ if $n$ is even, so the proof follows from Theorems 5.1, 4.4 and Corollary 3.1. □

In the next result, we describe the structure of $\Gamma(Q_n)$ for some values of $n$.

**Theorem 5.3.** *Let $n > 1$ be an integer.*

(i) *If $n = 2^\alpha$, $\alpha \geq 1$, then*

$$\Gamma(Q_n) \cong \begin{cases} K_4, & \text{if } \alpha = 1; \\ K_{\alpha+3} + \Gamma_N(D_n), & \text{otherwise.} \end{cases}$$

*where $\Gamma_N(D_n)$ is given by (4.16).*

(ii) *If $n = p^\alpha$, where $p$ is a prime, $p > 2, \alpha \geq 1$, then $\Gamma(Q_n) \cong K_{2\alpha+1} + \Gamma_N(D_n)$, where $\Gamma_N(D_n)$ given by (4.17).*

(iii) *If* $n = 2p$, *where* $p$ *is a prime,* $p > 2$, *then* $\Gamma(Q_n) \cong K_7 + \Gamma_N(D_n)$, *where* $\Gamma_N(D_n)$ *given by* (4.18).

(iv) *If* $n = pq$, *where* $p$, $q$ *are primes,* $2 < p < q$, *then* $\Gamma(Q_n) \cong K_7 + \Gamma_N(D_n)$, *where* $\Gamma_N(D_n)$ *given by* (4.19).

*Proof.* If $n = 2$, then $Q_n$ is dedekind with 4 proper subgroups and so $\Gamma(Q_n) \cong K_4$. The proofs of the remaining cases follows by Theorems 5.1, 5.2 and 4.5. □

**Theorem 5.4.** *Let* $n > 1$ *be an integer. If* $\tau(2n) + \sigma(n) < 2(n+1)$, *then* $\Gamma(Q_n)$ *is non-Hamiltonian.*

*Proof.* Let $S = V(\Gamma(Q_n)) - \mathcal{A}_1$. Since $\tau(2n) + \sigma(n) < 2(n+1)$, we have $|S| = \tau(2n) + \sigma(n) - n + 2 < n = |\mathcal{A}_1| = c(\Gamma(Q_n) - S)$. So by Theorem 2.1, $\Gamma(Q_n)$ is non-Hamiltonian.

□

**Theorem 5.5.**

(i) *If* $n = p^\alpha$, *where* $p$ *is a prime,* $\alpha \geq 1$, *then* $\Gamma(Q_n)$ *is Hamiltonian if and only if either* $p = 2$ *and* $\alpha \geq 1$ *or* $p = 3$ *and* $\alpha = 1$.

(ii) *If* $n = pq$, *where* $p$, $q$ *are distinct primes, then* $\Gamma(Q_n)$ *is Hamiltonian if and only if either* $p = 2$ *and* $q \leq 7$ *or* $p = 3$ *and* $q = 5$.

*Proof.* (i): Proof is divided into two cases.

**Case a.** $p = 2$. By Theorem 5.3(i), $\Gamma(D_n) \cong H + G$, where $H \cong K_{\alpha+3}$ and $G \cong \Gamma_N(D_n)$. Now by Theorem 4.7 and Lemma 4.1, $\Gamma(Q_n)$ is Hamiltonian.

**Case b.** $p > 2$.

If $\alpha = 1$, then by Theorem 5.3(ii), $\Gamma(Q_n) \cong H + \bigcup_{r=1}^{p} G_r$, where $H \cong K_3$ and for each $r = 1, 2, \ldots, p$, $G_r \cong K_1$. So by Lemma 4.1, $\Gamma(Q_n)$ is Hamiltonian if and only if $p = 3$.

If $\alpha \geq 2$, we take $S = V(\Gamma(Q_n)) - \mathcal{A}_1$. Then $|S| = \tau(2n) + \sigma(n) - n + 2 < n = |\mathcal{A}_1| = c(\Gamma(Q_n) - S)$. So by Theorem 2.1, $\Gamma(Q_n)$ is non-Hamiltonian.

(ii): We deal with the following cases:

**Case a.** $p = 2$. Then by Theorem 5.3(iii), $\Gamma(Q_n) \cong H + \bigcup_{r=1}^{q} G_r$, where $H \cong K_7$ and for each $r = 1, 2, \dots, q$, $G_r \cong K_3$. So by Lemma 4.1, $\Gamma(Q_n)$ is Hamiltonian if and only if $q \leq 7$.

**Case b.** $2 < p < q$. By Theorem 5.3(iv), $\Gamma(Q_n) \cong H + \Gamma_N(D_n)$, where $H \cong K_7$ and $\Gamma_N(D_n) \cong \bigcup_{i=1}^{p}\bigcup_{j=1}^{q} G_{ij}$, for each $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$ $G_{ij} \cong K_3$ with vertex set $\{u_i, v_j, w_{ij}\}$. Let the vertex set of $H$ be $\{N_i \mid i = 1, 2, \dots, k\}$.

If $p = 3$, $q = 5$, then $\Gamma_N(D_{15})$ is shown in Figure 1. It is easy to see that, $w_{11}-v_1-w_{21}-u_2-w_{22}-v_2- w_{12}-N_1-w_{23}- v_3-w_{13}-N_2- w_{24}-v_4-w_{14}- N_3-w_{25}-v_5- w_{35}-u_3-w_{32}-N_4 - w_{33} - N_5- w_{34} - N_6 - w_{31}- N_7 - w_{15} - u_1 - w_{11}$ is a spanning cycle in $\Gamma(Q_n)$.

If $p \geq 3$ and $q \geq 7$, we take $S = V(\Gamma(Q_n)) - \mathcal{A}_1$. Then $|S| = \tau(2n) + \sigma(n) - n + 2 < n = |\mathcal{A}_1| = c(\Gamma(Q_n) - S)$. So by Theorem 2.1, $\Gamma(Q_n)$ is non-Hamiltonian. □

# 6 Quasi-dihedral groups

For any positive integer $\alpha > 3$, the quasi-dihedral group of order $2^\alpha$, is defined by

$$QD_{2^\alpha} = \langle a, b \mid a^{2^{\alpha-1}} = b^2 = 1, bab^{-1} = a^{2^{\alpha-2}-1}\rangle.$$

The proper subgroups of $QD_{2^\alpha}$ are listed below:

(i) cyclic groups $H_0^r = \langle a^{\frac{2^{\alpha-1}}{r}}\rangle$, where $r$ is a divisor of $2^{\alpha-1}$, $r \neq 1$;

(ii) the dihedral group $H_1^{2^{\alpha-2}} = \langle a^2,\ b\rangle \cong D_{2^{\alpha-2}}$ and the dihedral subgroups $H_i^r$ of $H_1^{2^{\alpha-2}}$, where $r$ is a divisor of $2^{\alpha-2}$, $r \neq 2^{\alpha-2}$, $i = 1, 2, \dots, \frac{2^{\alpha-2}}{r}$;

(iii) the quaternion group $H_{2,2^{\alpha-3}} = \langle a^2, ab\rangle \cong Q_{2^{\alpha-3}}$ and the quaternion subgroups $H_{i,r}$ of $H_{2,2^{\alpha-3}}$, where $r$ is a divisor of $2^{\alpha-3}$, $r \neq 2^{\alpha-3}$, $i = 1, 2, \dots, \frac{2^{\alpha-3}}{r}$.

The only proper normal subgroups are listed in (i) together with $H_1^{2^{\alpha-2}}$, $H_{2,2^{\alpha-3}}$, $i = 1, 2$ of index 2. Thus

$$|L(QD_{2^\alpha})| = \alpha + 3.2^{\alpha-2} - 1, \tag{6.1}$$

and so

$$|V(\Gamma(QD_{2^\alpha}))| = \alpha + 3(2^{\alpha-2} - 1),$$

$$|V(\Gamma_N(QD_{2^\alpha}))| = 3.2^{\alpha-3} - 4.$$

## 6.1 Properties of $\Gamma_N(QD_{2^\alpha})$

First we describe the structure of $\Gamma_N(QD_{2^\alpha})$.

**Theorem 6.1.** *Let $\alpha > 3$ be an integer. Then*

$$\Gamma_N(QD_{2^\alpha}) \cong (K_2 + \Gamma_N(D_{2^{\alpha-2}})) \cup (K_2 + \Gamma_N(D_{2^{\alpha-3}})), \tag{6.2}$$

*where $\Gamma_N(D_{2^\alpha})$ is given by* (4.16).

*Proof.* The only non-normal subgroups of $QD_{2^\alpha}$ other than $\langle a \rangle$ are the dihedral subgroups of $H_1^{2^{\alpha-2}}$ and quaternion subgroups of $H_{2,2^{\alpha-3}}$. Here no non-normal subgroup of $H_1^{2^{\alpha-2}}$ permutes with any non-normal subgroup of $H_{2,2^{\alpha-3}}$. So $\Gamma_N(QD_{2^\alpha})$ is the disjoint union of $\mathcal{G}_1 \cup \mathcal{G}_2$, where $\mathcal{G}_1$ is the subgraph of $\Gamma_N(QD_{2^\alpha})$ induced by the dihedral subgroups of $H_1^{2^{\alpha-2}}$; and $\mathcal{G}_2$ is the subgraph of $\Gamma_N(QD_{2^\alpha})$ induced by the quaternion subgroups of $H_{2,2^{\alpha-3}}$. Here $H_1^{2^{\alpha-3}}$, $H_3^{2^{\alpha-3}}$ are normal subgroups of $H_1^{2^{\alpha-2}}$, but are not normal in $QD_{2^\alpha}$. Also $H_{2,2^{\alpha-4}}$, $H_{4,2^{\alpha-4}}$ are normal subgroups of $H_{2,2^{\alpha-3}}$, but are not normal in $QD_{2^\alpha}$. In view of these, it is easy to see that

$$\mathcal{G}_1 \cong K_2 + \Gamma_N(H_1^{2^{\alpha-2}}) \cong K_2 + \Gamma_N(D_{2^{\alpha-2}}),$$

and by Theorem 5.1,

$$\mathcal{G}_2 \cong K_2 + \Gamma_N(H_{2,2^{\alpha-3}}) \cong K_2 + \Gamma_N(Q_{2^{\alpha-3}}) \cong K_2 + \Gamma_N(D_{2^{\alpha-3}}).$$

Hence the proof. □

Next, we give the degrees of the vertices of $\Gamma_N(QD_{2^\alpha})$.

**Theorem 6.2.** *Let $\alpha > 3$ be an integer.*

(i) $\deg_{\Gamma_N(QD_{2^\alpha})}(H_i^r) = x_i^r - 2$, *for every divisor* $r$ *of* $2^{\alpha-2}$, $r \neq 2^{\alpha-2}$, $i = 1, 2, \ldots, \frac{2^{\alpha-2}}{r}$, *where* $x_i^r$ *is given by* (4.8).

(ii) $\deg_{\Gamma_N(QD_{2^\alpha})}(H_{i,r}) = x_i^r - 2$, *for every divisor* $r$ *of* $2^{\alpha-3}$, $r \neq 2^{\alpha-3}$, $i = 1, 2, \ldots, \frac{2^{\alpha-3}}{r}$, *where* $x_i^r$ *is given by* (4.8).

*Proof.* For every divisor $r$ of $2^{\alpha-2}$, $r \neq 2^{\alpha-2}$, $2^{\alpha-3}$, $i = 1, 2, \ldots, \frac{2^{\alpha-2}}{r}$, by Theorems 6.1 and 4.1, we have $\deg_{\Gamma_N(QD_{2^\alpha})}(H_i^r) = \deg_{\Gamma_N(D_{2^{\alpha-2}})}(H_i^r) + 2 = x_i^r - 4 + 2, = x_i^r - 2$, where $x_i^r$ is given in (4.8).

For $r = 2^{\alpha-3}$, $i = 1, 2$, we have $\deg_{\Gamma_N(QD_{2^\alpha})}(H_i^r) = x_i^r - |\{H_i^r, H_1^{2^{\alpha-2}}\}| = x_i^r - 2$, where $x_i^r$ is given in (4.8).

We can use the similar argument to prove the part (ii) of this result, in view of Theorem 5.1. □

**Corollary 6.1.** *Let* $\alpha > 3$ *be an integer. Then*

$$|E(\Gamma_N(QD_{2^\alpha}))| = 2^{\alpha-3}(12\alpha - 49) + 14.$$

*Proof.* By Theorem 6.1,

$$\begin{aligned} |E(\Gamma_N(QD_{2^\alpha}))| &= |E(K_2 + \Gamma_N(D_{2^{\alpha-2}}))| + |E(K_2 + \Gamma_N(D_{2^{\alpha-3}}))| \\ &= 2|V(\Gamma_N(D_{2^{\alpha-2}}))| + 1 + |E(\Gamma_N(D_{2^{\alpha-2}}))| \\ &\quad + 2|V(\Gamma_N(D_{2^{\alpha-3}}))| + 1 + |E(\Gamma_N(D_{2^{\alpha-3}}))|. \end{aligned} \tag{6.3}$$

Now by applying Corollary 4.1 and (4.2), in (6.3), the result follows. □

**Corollary 6.2.** *Let* $\alpha > 3$ *be an integer.*

(i) $\alpha(\Gamma_N(QD_{2^\alpha})) = 3.2^{\alpha-4}$.

(ii) $\omega(\Gamma_N(QD_{2^\alpha})) = 2(\alpha - 2)$.

(iii) $\chi(\Gamma_N(QD_{2^\alpha})) = 2(\alpha - 2)$.

(iv) $\Gamma_N(QD_{2^\alpha})$ *is weakly perfect.*

(v) $\gamma(\Gamma_N(QD_{2^\alpha})) = 2$

(vi) $\Gamma_N(QD_{2^\alpha})$ *is non-Eulerian.*

(vii) $\Gamma_N(QD_{2^\alpha})$ *is non-Hamiltonian.*

*Proof.* (i): Using Theorem 6.1,

$$\begin{aligned}\alpha(\Gamma_N(QD_{2^\alpha})) &= \alpha(K_2 + \Gamma_N(D_{2^{\alpha-2}})) + \alpha(K_2 + \Gamma_N(D_{2^{\alpha-3}}))\\ &= \alpha(\Gamma_N(D_{2^{\alpha-2}})) + \alpha(\Gamma_N(D_{2^{\alpha-3}})).\end{aligned}$$

So the proof follows by Theorem 4.4(i).

(ii): Using Theorem 6.1,

$$\begin{aligned}\omega(\Gamma_N(QD_{2^\alpha})) &= \max\{\omega(K_2 + \Gamma_N(D_{2^{\alpha-2}})), \omega(K_2 + \Gamma_N(D_{2^{\alpha-3}}))\}\\ &= \max\{2 + \omega(\Gamma_N(D_{2^{\alpha-2}})), 2 + \omega(\Gamma_N(D_{2^{\alpha-3}}))\}.\end{aligned}$$

So the proof follows by Theorem 4.4(ii).

(iii): Using Theorem 6.1,

$$\begin{aligned}\chi(\Gamma_N(QD_{2^\alpha})) &= \max\ \{\chi(K_2 + \Gamma_N(D_{2^{\alpha-2}})), \chi(K_2 + \Gamma_N(D_{2^{\alpha-3}}))\}\\ &= \max\ \{2 + \chi(\Gamma_N(D_{2^{\alpha-2}})), 2 + \chi(\Gamma_N(D_{2^{\alpha-3}}))\}\end{aligned}$$

So the proof follows by Theorem 4.4(iii).

(iv): Follows from (ii) and (iii).

(v): Follows from Theorem 6.1.

(vi)-(vii): Since by Theorem 6.1, $\Gamma_N(QD_{2^\alpha})$ is disconnected, so the proof follows. □

## 6.2 Properties of $\Gamma(QD_{2^\alpha})$

In the following result we describe the structure of $\Gamma(QD_{2^\alpha})$

**Theorem 6.3.** *Let $\alpha > 3$ be an integer. Then $\Gamma(QD_{2^\alpha}) \cong K_{\alpha+1} + \Gamma_N(QD_{2^\alpha})$, where $\Gamma_N(QD_{2^\alpha})$ is given by* (6.1)

*Proof.* Since the number of proper normal subgroups of $QD_{2^\alpha}$ is $\alpha+1$. So by Theorems 3.1 and 6.1, the proof follows. $\square$

**Corollary 6.3.** *Let $\alpha > 3$ be an integer.*

(1) $\deg_{\Gamma(QD_{2^\alpha})}(H_0^r) = \alpha + 3.2^{\alpha-2} - 4$, *for every divisor $r$ of $2^{\alpha-1}$, $r \neq 1$.*

(2) $\deg_{\Gamma(QD_{2^\alpha})}(H_1^{2^{\alpha-2}}) = \alpha + 3.2^{\alpha-2} - 4 = \deg_{\Gamma(QD_{2^\alpha})}(H_2^{2^{\alpha-3}})$.

(3) $\deg_{\Gamma(QD_{2^\alpha})}(H_i^r) = \alpha + x_i^r - 1$, *for every divisor $r$ of $2^{\alpha-2}$, $r \neq 2^{\alpha-2}$, $i = 1, 2, \dots, \frac{2^{\alpha-2}}{r}$, where $x_i^r$ is given by* (4.8).

(4) $\deg_{\Gamma(QD_{2^\alpha})}(H_{i,r}) = \alpha + x_i^r - 1$, *for every divisor $r$ of $2^{\alpha-3}$, $r \neq 2^{\alpha-3}$, $i = 1, 2, \dots, \frac{2^{\alpha-3}}{r}$, where $x_i^r$ is given by* (4.8).

*Proof.* Follows by Corollary 3.1, Theorem 6.2 and (6.1) and from the fact that $H_1^{2^{\alpha-2}}$ and $H_2^{2^{\alpha-3}}$ are normal in $QD_{2^\alpha}$. $\square$

**Corollary 6.4.** *Let $\alpha > 3$ be an integer. Then*

$$|E(\Gamma(QD_{2^\alpha}))| = \frac{1}{2}\left\{2^{\alpha-2}(18\alpha - 43) + \alpha^2 - 7\alpha + 20\right\}.$$

*Proof.* Follows by Corollaries 6.1, 3.1, (6.1) and from the fact that the number of proper normal subgroups of $QD_{2^\alpha}$ is $\alpha + 1$. $\square$

**Corollary 6.5.** *Let $\alpha > 3$ be an integer.*

(i) $\alpha(\Gamma(QD_{2^\alpha})) = 3.2^{\alpha-4}$.

(ii) $\omega(\Gamma(QD_{2^\alpha})) = 3(\alpha - 1)$.

(iii) $\chi(\Gamma(QD_{2^\alpha})) = 3(\alpha - 1)$.

(iv) $\Gamma(QD_{2^\alpha})$ *is weakly perfect.*

(v) $\Gamma(QD_{2^\alpha})$ *is non-Eulerian.*

(vi) $\Gamma(QD_{2^\alpha})$ *is Hamiltonian.*

*Proof.* (i)-(iii): Follows by the part (iii), (iv), and (v) of Corollary 3.1 and by parts (i), (ii), (iii) of Corollary 6.2.

(iv): Follows by (ii) and (iii).

(v): By Corollary 6.3(ii), if $\alpha$ is odd, then $\deg_{\Gamma(QD_{2^\alpha})}(H_i^r)$ is odd for every divisor $r$ of $2^{\alpha-2}$, and by Corollary 6.3(i), if $\alpha$ is even, then $\deg(H_0^r)$ is odd for every divisor $r$ of $2^{\alpha-1}$. So the proof follows.

(vi): By Theorems 6.1 and 6.3, $\Gamma(QD_{2^\alpha}) = H + G_1 \cup G_2$, where $H \cong K_{\alpha+1}$, $G_1 \cong K_2 + \Gamma_N(D_{2^{\alpha-2}})$ and $G_2 \cong K_2 + \Gamma_N(D_{2^{\alpha-3}})$. Now, by Corollary 4.7, $G_1$ and $G_2$ contains a Hamiltonian path. So by Lemma 4.1, $\Gamma(QD_{2^\alpha})$ is Hamiltonian. $\square$

# 7 Modular groups

For any integer $\alpha > 2$ and any prime $p$, the modular group $M_{p^\alpha}$ of order $p^\alpha$ is defined by

$$M_{p^\alpha} = \langle a, b | \; a^{p^{\alpha-1}} = b^p = 1, bab^{-1} = a^{p^{\alpha-2}-1} \rangle.$$

If $p^\alpha = 8$, then $M_8 \cong D_4$ and its corresponding permutability graphs are given by Theorem 4.5(i) and Corollary 4.8(i). If $p^\alpha \neq 8$, then the subgroup lattice of $M_{p^\alpha}$ is isomorphic to that of $\mathbb{Z}_{p^{\alpha-1}} \times \mathbb{Z}_p$ and it is shown in [6, p. 210, Figure 4]. Also if $p^\alpha \neq 8$, then the only proper normal subgroups of $M_{p^\alpha}$ are $\langle a^{p^{\alpha-2}} \rangle$, $\langle a^i b^j \rangle$ and $\langle a^k, b \rangle$, $i = 1, 2, \ldots, p^{\alpha-3}, j = 0, 1, \ldots, p-1$ and $k = 1, 2, \ldots, p^{\alpha-2}$. Hence

$$|L(M_{p^\alpha})| = \begin{cases} 4, & \text{if } p^\alpha = 8; \\ (\alpha-1)(p+1)+2, & \text{otherwise.} \end{cases}$$

and so

$$|V(\Gamma(M_{p^\alpha}))| = \begin{cases} 8, & \text{if } \; p^\alpha = 8; \\ (\alpha-1)(p+1), & \text{otherwise.} \end{cases}$$

$$|V(\Gamma_N(M_{p^\alpha}))| = \begin{cases} 4, & \text{if } p^\alpha = 8; \\ p, & \text{otherwise.} \end{cases}$$

It is well known that if $p^\alpha \neq 8$, then any two subgroups of $M_{p^\alpha}$ permutes with each other.

Combining all these facts together, we have the following result.

**Theorem 7.1.** *Let $\alpha > 3$ be an integer.*

(i) $\Gamma_N(M_{p^\alpha}) \cong \begin{cases} 2K_2, & \textit{if } \ p^\alpha = 8; \\ K_p, & \textit{otherwise.} \end{cases}$

(ii) $\Gamma(M_{p^\alpha}) \cong \begin{cases} K_4 + 2K_2, & \textit{if } \ p^\alpha = 8; \\ K_{(\alpha-1)(p+1)}, & \textit{otherwise.} \end{cases}$

Since $\Gamma_N(M_{p^\alpha})$ and $\Gamma(M_{p^\alpha})$ are complete graphs if $p^\alpha \neq 8$, so one can easily obtain the other properties of these graphs.

# 8 Conclusions

In this paper, we have studied the structure and some properties of permutability graphs of subgroups and permutability graph of non-normal subgroups of the groups $D_n$, $Q_n$, $QD_{2^\alpha}$ and $M_{p^\alpha}$. In particular, we showed that the structure of $\Gamma(D_n)$, $\Gamma_N(Q_n)$, $\Gamma(Q_n)$, $\Gamma_N(QD_{2^\alpha})$ and $\Gamma(QD_{2^\alpha})$ depends on the structure of $\Gamma_N(D_n)$. In this sequel, in Theorems 6.1 and 6.3, we explicitly described structure of $\Gamma_N(QD_{2^\alpha})$ and $\Gamma(QD_{2^\alpha})$ respectively. For the values of $n$ other than that mentioned in Theorem 4.5, the structure of the graph $\Gamma_N(D_n)$ becomes complicated and so further investigation of the structure of $\Gamma_N(D_n)$ is essential to study the further properties of these remaining graphs.

Also, in Theorems 4.6, 4.10, 4.11, 5.4, 5.5 and in Corollary 4.3, we have discussed the Hamiltonicity of $\Gamma_N(D_n)$, $\Gamma(D_n)$, $\Gamma(Q_n)$ for some values of $n$. Now we pose the following:

**Problem 8.1.** *Determine the values of $n$, for which $\Gamma_N(D_n)$, $\Gamma(D_n)$, $\Gamma(Q_n)$ are Hamiltonian.*

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